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Spinor Representation of Electromagnetic Fields on Noncommutative Spaces

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We apply Campolattaro's spinor representation of the electromagnetic field to noncommutative spaces. The spinor representation of the (self-dual) electromagnetic field on noncommutative spaces is obtained.

KEY WORDS: noncommutative space; electromagnetic field; spinor.

1. INTRODUCTION

It is well known that the electromagnetic field can be written in a number of different forms. Campolattaro (1980a,b) started with the analysis of the Maxwell equations by writing the electromagnetic field tensor in the spinor form, using the standard Dirac γ-matrices. Vaz and Rodrigues (1993, 1997) expressed the spinor representation of electromagnetic field in the Clifford bundle formalism.

The spinor representation of electromagnetic fields plays an important role in our understanding of the structure of space–time. One of us (Hu and Hu, 1998) suggested the relation between the Campolattaro's formalism and the Witten's monopole equations (Witten, 1994). See also Vaz (1997).

On the other hand, the field theories on noncommutative spaces are of great interest now due to the recent development of the superstring theory. It was shown that in the presence of a background Neveu-Schwarz B-field, the gauge theory living on D-branes becomes noncommutative (Connes *et al*., 1998). On the basis of existence of the different regularization procedures in string theory, Seiberg and Witten (1999) claimed that certain noncommutative gauge theories are equivalent to commutative ones. In particular, they argued that there exists a map from a commutative gauge field to a noncommutative one, which is compatible with the gauge structure of each. This map has become known as the Seiberg–Witten map.

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For the discussions of noncommutative U (1) gauge theory (Maxwell theory), see, e.g., Martin and Sanchez-Ruiz (1999), Hayakawa (2000), Sheikh-Jabbari (2000), Bichl *et al*. (2001), Cai (2001), Chaichian *et al*. (2001), Guralnik *et al*. (2001), Okawa and Ooguri (2001), Grimstrup *et al*. (2001).

In this paper, we apply Campolattaro's formulation of the electromagnetic field tensor $F^{\mu\nu}$ in the bilinear form $F^{\mu\nu} = \overline{\Psi} S^{\mu\nu} \Psi$ to noncommutative spaces. The spinor representation of the electromagnetic field on noncommutative spaces is obtained.

2. CAMPOLATTARO'S SPINOR REPRESENTATION OF ELECTROMAGNETIC FIELDS

Campolattaro (1980a,b) started with the analysis of the Maxwell equations by writing the electromagnetic field tensor $F^{\mu\nu}$ in the equivalent bilinear form

$$F^{\mu\nu} = \overline{\Psi} S^{\mu\nu} \Psi, \tag{1}$$

where $\mu, \nu = 0, 1, 2, 3$. Ψ is a Dirac spinor, and $\overline{\Psi} = \Psi^{\dagger}\gamma^0$ is the Dirac conjugation of Ψ. $S^{\mu\nu}$ is the spin operater defined by

$$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}], \tag{2}$$

and the γ's are the Dirac matrices satisfying

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}, \tag{3}$$

with $\eta^{\mu\nu}$, the Minkowski metric tensor, given by $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

In this representation the dual tensor $\tilde{F}^{\mu\nu}$ is given by

$$\tilde{F}^{\mu\nu} = \overline{\Psi}\gamma^5 S^{\mu\nu}\Psi. \tag{4}$$

From now on, the Einstein sum convention is adopted throughout. The Maxwell equations read (a comma followed by an index represents the partial derivative with respect to the variable with that index)

$$(\overline{\Psi} S^{\mu\nu}\Psi)_{,\mu} = j^{\nu}, \tag{5}$$

$$(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)_{,\mu} = 0. \tag{6}$$

Moreover, the duality (Misner and Wheeler, 1957; Rainich, 1925) by the complexion α, namely

$$\overline{F}^{\mu\nu} = F^{\mu\nu}\cos\alpha + \tilde{F}^{\mu\nu}\sin\alpha, \tag{7}$$

is equivalent to a Touschek–Nishijima transformation (Nishijima, 1957; Touschek, 1957) for the spinor Ψ to the spinor Ψ' given by

$$\Psi' = e^{\gamma^5\alpha/2}\Psi, \tag{8}$$

with

$$e^{\gamma^5\alpha} = \cos\alpha + \gamma^5 \sin\alpha, \tag{9}$$

and

$$\cos\alpha = \frac{\overline{\Psi}\Psi}{\rho}, \tag{10}$$

$$\sin\alpha = \frac{\overline{\Psi}\gamma^5\Psi}{\rho}, \tag{11}$$

ρ being the positive square root of

$$\rho^2 = (\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2. \tag{12}$$

Campolattaro showed that the two spinor Maxwell equations (5) and (6) are equivalent to a single nonlinear first-order equation for the spinor, namely

$$\gamma^\mu \Psi_{,\mu} = -i\gamma^\mu \frac{e^{\gamma^5\alpha}}{\rho} \left\{\mathrm{Im}(\overline{\Psi}_{,\mu}\Psi) - j_\mu - \gamma^5\, \mathrm{Im}\big(\overline{\Psi}_{,\mu}\gamma^5\Psi\big)\right\} \Psi. \tag{13}$$

3. SPINOR REPRESENTATION OF ELECTROMAGNETIC FIELDS ON NONCOMMUTATIVE SPACES

3.1. Moyal * Product

We consider a noncommutative space with coordinates $\hat{x}^i$ characterized by the algebra

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{14}$$

where θ^{ij} is an antisymmetric constant tensor with $\theta^{ij} = -\theta^{ji}$. Field theories in such a space can be realized as a deformation of the usual field theory in an ordinary (commutative) space by changing the product of two fields to the Moyal $*$ product (Moyal, 1949) defined by

$$f(x) * g(x) = \exp\left(\frac{i}{2}\theta^{kl}\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^l}\right) f(y)g(z)|_{y=z=x}. \tag{15}$$

Note that the first term on the right side gives the ordinary product. Also the commutator (14) is realized as

$$[x^i, x^j]_* \equiv x^i * x^j - x^j * x^i = i\theta^{ij}. \tag{16}$$

We also consider the case of matrix-valued functions f and g. In this case, we define the Moyal $*$ product to be the tensor product of matrix multiplication with the $*$ product of functions as just defined. The extended $*$ product is still associative.

3.2. Seiberg–Witten Map

Let $\hat{A}_i$ be a noncommutative gauge field on a noncommutative space, whose coordinates obey $[x^i, x^j]_* = i\theta^{ij}$. Denote by A_i the counterpart of $\hat{A}_i$, the ordinary gauge field on the ordinary space. The map between A_i and $\hat{A}_i$, called the Seiberg–Witten map (Seiberg and Witten, 1999), is characterized by the differential equation with respect to θ,

$$\delta\hat{A}_i(\theta) = -\frac{1}{4}\delta\theta^{jk}[\hat{A}_j * (\partial_k\hat{A}_i + \hat{F}_{ki}) + (\partial_k\hat{A}_i + \hat{F}_{ki}) * \hat{A}_j], \tag{17}$$

with the initial condition

$$\hat{A}_i(\theta = 0) = A_i. \tag{18}$$

Here $*$ is the Moyal $*$ product. The field strength $\hat{F}_{ij}$ is defined as

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i - i\hat{A}_i * \hat{A}_j + i\hat{A}_j * \hat{A}_i. \tag{19}$$

The differential equation (17) is known as the Seiberg–Witten equation (Seiberg and Witten, 1999). The theory obtained in this way reduces to conventional $U(N)$ Yang–Mills theory for $\theta \to 0$. The rank 1 case ($N = 1$) corresponds to the Abelian U (1) gauge theory (Maxwell electromagnetic theory) for $\theta = 0$.

For the rank 1 case, the gauge transformation of $\hat{A}_i(x)$ is given by

$$\hat{A}'_i(x) = U(x) * \hat{A}_i(x) * U^{-1}(x) - iU(x) * \partial_i U^{-1}(x), \tag{20}$$

where $U(x) = (e*)^{i\lambda(x)}$ for real functions $\lambda(x)$, and the $(e*)$ is defined by the usual Taylor expansion, with all products of λ's replaced by the $*$ ones. One can find that $U^{-1} = (e*)^{-i\lambda(x)}$ satisfies $U^{-1} * U = 1$. It should be noticed that the non-Abelian character of the above gauge transformations is due to the noncommutativity of the space.

3.3. Spinor Representation of Electromagnetic Fields on Noncommutative Spaces

Naively, to get a physical quantity on a noncommutative space, we simply take this quantity on the corresponding commutative space and replace all products by the $*$ products.

We now introduce the spinor representation of the electromagnetic field on the noncommutative space–time V with coordinates x^μ characterized by

$$[x^\mu, x^\nu]_* = i\theta^{\mu\nu}.$$

Let $\hat{F}^{\mu\nu}$ be the noncommutative electromagnetic field on V. $\hat{F}^{\mu\nu}$ can be written in the following form:

$$\hat{F}^{\mu\nu} = \overline{\hat{\Psi}} S^{\mu\nu} * \hat{\Psi}, \tag{21}$$

where $\mu,\ \nu = 0,\ 1,\ 2,\ 3$. $\hat{\Psi}$ is a Dirac spinor on V with the initial condition $\hat{\Psi}(\theta = 0) = \Psi$. The constant Dirac matrices γ's are the same as in the commutative case. $\overline{\hat{\Psi}} = \hat{\Psi}^{\dagger}\gamma^{0}$ is the Dirac conjugation of $\hat{\Psi}$. The spin operator $S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$ is the same as in Section 2.

Notice that (21) can also be written as

$$\hat{F}^{\mu\nu} = \overline{\hat{\Psi}} * S^{\mu\nu}\Psi.$$

In this spinor representation the noncommutative counterpart of the dual tensor $\tilde{F}^{\mu\nu}$ is given by

$$\hat{\tilde{F}}^{\mu\nu} = \overline{\hat{\Psi}}\gamma^{5}S^{\mu\nu} * \hat{\Psi}. \tag{22}$$

From $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, where $\epsilon^{\mu\nu\alpha\beta}$ is an antisymmetric Levy–Civita tensor with $\epsilon^{1234} = -i$, one has

$$\hat{\tilde{F}}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\hat{F}_{\alpha\beta} = \tilde{\hat{F}}^{\mu\nu}. \tag{23}$$

Therefore, Eq. (22) can be rewritten as

$$\tilde{\hat{F}}^{\mu\nu} = \overline{\hat{\Psi}}\gamma^{5}S^{\mu\nu} * \hat{\Psi}. \tag{24}$$

The noncommutative Maxwell equations read

$$(\overline{\hat{\Psi}}S^{\mu\nu} * \hat{\Psi})_{,\mu} = \hat{j}^{\nu}, \tag{25}$$

$$(\overline{\hat{\Psi}}\gamma^{5}S^{\mu\nu} * \hat{\Psi})_{,\mu} = 0. \tag{26}$$

Here $\hat{j}^{\nu}$ is the noncommutative counterpart of j^{ν}.

Denote $\epsilon = \rho^{2} = (\overline{\Psi}\Psi)^{2} + (\overline{\Psi}\gamma^{5}\Psi)^{2}$, and $\kappa = \overline{\Psi}\Psi + \gamma^{5}\overline{\Psi}\gamma^{5}\Psi$. The noncommutative counterpart ϵ is given by

$$\hat{\epsilon} = (\overline{\hat{\Psi}} * \hat{\Psi})^{2} + (\overline{\hat{\Psi}}\gamma^{5} * \hat{\Psi})^{2}. \tag{27}$$

The noncommutative counterpart of κ reads

$$\hat{\kappa} = \overline{\hat{\Psi}} * \hat{\Psi} + \gamma^{5}\overline{\hat{\Psi}}\gamma^{5} * \hat{\Psi}. \tag{28}$$

It follows that the noncommutative Maxwell equations are equivalent to the following equation:

$$\gamma^{\mu}\hat{\epsilon} * \hat{\Psi}_{,\mu} = -i\gamma^{\mu}\hat{\kappa} * \left\{\mathrm{Im}(\overline{\hat{\Psi}}_{,\mu} * \hat{\Psi}) - \hat{j}_{\mu} - \gamma^{5}\,\mathrm{Im}\left(\overline{\hat{\Psi}}_{,\mu}\gamma^{5} * \hat{\Psi}\right)\right\} * \hat{\Psi}. \tag{29}$$

Remark. The noncommutativity breaks the symmetry. The term $\frac{e^{\gamma^{5}\alpha}}{\rho}$ in Eq. (13) does not exist in the above equation. Instead, it generates $\hat{\epsilon}$ and $\hat{\kappa}$ on the two sides of the equation, respectively.

3.4. Spinor Representation of Self-Dual Electromagnetic Fields on Noncommutative Spaces

From Eqs. (21) and (24), one has

$$\begin{aligned}\hat{F}^{+}_{\mu\nu} &= \frac{1}{2}(\hat{F}_{\mu\nu} + \tilde{\hat{F}}_{\mu\nu}) \\ &= \overline{\hat{\Psi}}\frac{1+\gamma^5}{2}S_{\mu\nu} * \hat{\Psi}.\end{aligned}$$

Denote $\frac{1+\gamma^5}{2}\hat{\Psi} = \hat{\Phi}$; then the self-dual part of the electromagnetic field on V is given by

$$\hat{F}^{+}_{\mu\nu} = \overline{\hat{\Phi}}S_{\mu\nu} * \hat{\Phi}. \tag{30}$$

From Eqs. (25) and (26), one has

$$\partial^{\mu}\hat{F}^{+}_{\mu\nu} = \frac{1}{2}\hat{j}_{\nu}.$$

Equation (30) has the equivalent form

$$\mathrm{Im}\Big(\overline{\hat{\Phi}}\gamma^{\mu}\gamma^{\nu} * \partial_{\nu}\hat{\Phi}\Big) + \mathrm{Im}(\hat{\Phi} * \partial^{\mu}\hat{\Phi}) + \frac{1}{2}\hat{j}^{\mu} = 0. \tag{31}$$

One can verify that the positive chirality spinor $\hat{\Phi}$ satisfies the following equation:

$$\gamma^{\mu}(\overline{\hat{\Phi}} * \hat{\Phi}) * \hat{\Phi}_{,\mu} = i\left\{\mathrm{Im}(\overline{\hat{\Phi}} * \hat{\Phi}_{,\mu}) + \frac{1}{2}\hat{j}_{\mu}\right\} * \gamma^{\mu}\hat{\Phi}. \tag{32}$$

4. THE CASE OF REVISED MAXWELL EQUATIONS

Campolattaro (1990a,b) assumed that together with an electric current j_{μ}, there also exists a magnetic monopole current g_{μ}. Maxwell equations read

$$F^{\mu\nu}_{,\mu} = j^{\nu}, \tag{33}$$

$$\tilde{F}^{\mu\nu}_{,\mu} = g^{\nu}. \tag{34}$$

There exists a spinor such that

$$F^{\mu\nu} = \overline{\Psi}S^{\mu\nu}\Psi, \tag{35}$$

$$\tilde{F}^{\mu\nu} = \overline{\Psi}\gamma^5 S^{\mu\nu}\Psi. \tag{36}$$

It was shown that the spinor equation (13) in the presence of magnetic monopoles reads

$$\gamma^{\mu}\Psi_{,\mu} = -i\gamma^{\mu}\frac{e^{\gamma^5\alpha}}{\rho}\left\{\mathrm{Im}(\overline{\Psi}_{,\mu}\Psi) - j_{\mu} - \gamma^5\left[\mathrm{Im}(\overline{\Psi}_{,\mu}\gamma^5\Psi) - g_{\mu}\right]\right\}\Psi. \tag{37}$$

It is straightforward to see that the noncommutative Maxwell equations in the presence of magnetic monoples read

$$\hat{F}^{\mu\nu}_{,\mu} = \hat{j}^{\nu}, \tag{38}$$

$$\tilde{\hat{F}}^{\mu\nu}_{,\mu} = \hat{g}^{\nu}. \tag{39}$$

There exists a spinor $\hat{\Psi}$ such that

$$\hat{F}^{\mu\nu} = \overline{\hat{\Psi}} S^{\mu\nu} * \hat{\Psi}, \tag{40}$$

$$\tilde{\hat{F}}^{\mu\nu} = \overline{\hat{\Psi}} \gamma^5 S^{\mu\nu} * \hat{\Psi}. \tag{41}$$

One can find that the spinor equation (29) in the presence of magnetic monopoles reads

$$\gamma^{\mu}\hat{\epsilon} * \hat{\Psi}_{,\mu} = -i\gamma^{\mu}\hat{\kappa} * \left\{\mathrm{Im}(\overline{\hat{\Psi}}_{,\mu} * \hat{\Psi}) - \hat{j}_{\mu} - \gamma^5\left[\mathrm{Im}\left(\overline{\hat{\Psi}}_{,\mu}\gamma^5 * \hat{\Psi}\right) - \hat{g}_{\mu}\right]\right\} * \hat{\Psi}. \tag{42}$$

From Eqs. (38) and (39), one has

$$\partial^{\mu}\hat{F}^{+}_{\mu\nu} = \frac{1}{2}(\hat{j}_{\nu} + \hat{g}_{\nu}).$$

From Eqs. (40) and (41), we have

$$\hat{F}^{+}_{\mu\nu} = \overline{\hat{\Phi}} S_{\mu\nu} * \hat{\Phi}. \tag{43}$$

One can also find that the positive chirality spinor $\hat{\Phi}$ in the presence of magnetic monopoles satisfies the following equation:

$$\gamma^{\mu}(\overline{\hat{\Phi}} * \hat{\Phi}) * \hat{\Phi}_{,\mu} = i\left\{\mathrm{Im}(\overline{\hat{\Phi}} * \hat{\Phi}_{,\mu}) + \frac{1}{2}(\hat{j}_{\mu} + \hat{g}_{\mu})\right\} * \gamma^{\mu}\hat{\Phi}. \tag{44}$$

5. DISCUSSION

We have proposed the spinor representation of the electromagnetic field on the noncommutative space–time. The spinor equations we obtained include the higher derivatives. This leads to the nonlocal interactions of the fields.

In the Minkowski space–time, one can find that the Maxwell equations in the spinor form are not equivalent to the Dirac equation (Gsponer, 2002). In the corresponding noncommutative case, this claim is also true. Nevertheless, the spinor representation of the electromagnetic field is important. For example, it provides a powerful tool for us to study the topology of four-dimensional differential manifolds (Witten, 1994; Hu and Hu, 1998).

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